Minority Game: a mean-field-like approach

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Abstract

We calculate the standard deviation of $(N_1 - N_0)$, the difference of the number of agents choosing between the two alternatives of the minority game. Our approach is based on two approximations: we use the whole set of possible strategies, rather than only those distributed between the agents involved in a game; moreover, we assume that a period-two dynamics discussed by previous authors is appropriate within the range of validity of our work. With these appproximations we introduce a set of states of the system, and are able to replace time averages by ensemble averages over these states. Our results show a very good agreement with simulations results for most part of the informationally efficient phase.

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I. INTRODUCTION

The minority game (MG) is an adaptive game introduced by Challet and Zhang [1] to study competitive systems whose available resources are finite (ecology, financial systems, traffic in Internet, etc).

At every step of the game, the N participants (agents) must choose one of two alternatives (0 or 1, to buy or to sell, to choose one of two possible routes, etc.), and the winners will be those who turn out to be in the minority. Each agent decides her move based on the *history* of the game (a string of m bits with the information about the m previous minority sides), using one of her set of s prescriptions or *strategies*. Feedback is established by a reward system whereby every winner agent gets a point; moreover, out of all the strategies in the game, those that correctly predicted the minority side also get a (so-called virtual) point. The game is adaptive, because an agent will make her choise using the strategy that at this particular time has more points.

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For every time step, let us call N_0 (N_1) the number of agents choosing side 0 (1), such that $N_0 + N_1 = N$. The main variable that is usually considered is σ, the standard deviation of the difference $(N_1 - N_0)$, as a function of N, m, and s,

$$\sigma^2 = \frac{1}{T} \sum_{t=1}^{T} (N_1 - N_0)^2 \tag{1}$$

where T is the number of steps of the game.

The standard deviation σ is a measure of the form in which the resources (points) of the population are used: whenever $|N_1 - N_0|$ is small, more agents get a point, implying a better utilization of the resources. The model has attracted some attention because for certain values of m, s and N, σ turns out to be smaller than σ_r, the standard deviation of a random-choise game, where the N agents choose sides randomly. This result implies that althought there is no agent-agent interaction in the MG, there is some kind of emergent coordination; in other words, there is an effective interaction that appears through global magnitudes of the system. This collective behavior motivated a diversity of studies of the MG, with a variety of techniques: numerical simulations [1], [2]; mean-field approximations [3]; equivalence with spin-glass models [4]; thermal treatments [5], etc. Manuca et al. [3] have studied the complete succession of minority sides, and used a specific mean-field to calculate σ. In the present work we calculate σ for the same set of values of N, m and s used by these authors, obtaining a somewhat better agreement with numerical results.

The public information on which strategies are based is the record of the last m minority sides (that is why m is called the *memory* of the game). Then, there are $\mathcal{H} = 2^m$ different histories, the history at time t being the string $k = (\chi_{t-m}, ..., \chi_{t-1})$ where χ_t is the minority side (0 or 1) which results after time t. In the following we will write k as an integer between 0 and $2^m - 1$. Every strategy is a function that assigns *outputs* (predictions of the following minority side) for every one of the $\mathcal{H}$ possible *inputs* . Hence, there are $\mathcal{L} = 2^{\mathcal{H}}$ different strategies in a game of memory m ; at the beginning of the game each agent is randomly assigned s strategies, from the pool of $\mathcal{L}$ (with replacement).

It is known [3] that data for σ^2/N displays scaling for every value of s when plotted as a function of $z = 2^m/N$. In fact the plot of σ^2/N vs z has two different regions, as it is shown in Fig. 1 : *(i)* $z \lesssim 0.5$, which is an informationally efficient phase. Here σ^2/N is a decreasing function of z, showing a change of behavior around $\sigma^2/N \approx 1$; *(ii)* $z > 0.5$, where σ^2/N increases and asymptotically approaches the line $\sigma^2/N = 1$. This line corresponds to the random-choise game. Our calculation is appropriate within a good part of the first region.

Usually, numerical simulations are carried over for a game of T time steps, and afterwards the results are averaged over several independent runs of the game. In every run there are $\ell \equiv sN$ strategies distributed between the agents, out of the pool $\mathcal{L}$. There are two points worth noticing: the pool $\mathcal{L}$ of *all* the strategies is *symmetrical* , in the sense that for every possible history, the number of strategies predicting $\chi_t = 1$ is equal to the number of strategies predicting $\chi_t = 0$; on the other hand, every set of ℓ strategies used in each run is only *approximately* symmetrical, because it is just a finite sample of $\mathcal{L}$. When the sample is only of a moderate size, however, the difference between the properties of the sample and the pool is small, and the corresponding uncertainty is known. Indeed, this is widely used in standard 'sampling' techniques [6]: for instance, the uncertainty associated with samples

of size ≈ 400 is less than $\approx 5\%$. These considerations are at the base of our calculation, as we now explain.

In the following we will make analytic calculations over the whole pool $\mathcal{L}$, rather than over samples of size ℓ. In this form we will benefit from its symmetry, and also will be able to calculate several magnitudes by simply counting different sets of pairs of strategies (for $s = 2$). In this sense we are using sampling techniques the other way around: we start from the pool, to find the samples' properties. As it will be clear below, this is a mean-field-like approach.

Equation (1) calculates σ as a time average over T time steps. We will replace this by an ensamble average over a (restricted) set of states of the system. In Sec. II we introduce the states and describe how a *period-two dynamics* observed in Ref. [3] can be used to substantially reduce the number of states needed. In Sec. III we write down σ in terms of the new variables; to carry out the actual calculation of σ we found it useful to make use of some ad-hoc diagrams that are also explained. In the final section we offer some conclusions.

II. THE SPACE OF AVAILABLE STATES

In this work we will only consider games where $s = 2$, *i.e.* every agent has two strategies to choose from. To find σ from Eq. (1) we need to write down expressions for N_1/N and N_0/N. It is clear that the values of these ratios will fluctuate with the actual numerical realization. We now *assume* that we can obtain approximate values for these variables by using not N, the actual number of agents, but rather a game with $\mathcal{N}$, the maximum amount of (virtual) agents that can originate in the pool. $\mathcal{N}$ is equal to $\binom{\mathcal{L}}{2} + \mathcal{L}$, the number of pairs of strategies that it is possible to make from the pool $\mathcal{L}$ (with replacement) [1] . Within this approximation, we will calculate N_1/N as $\approx \mathcal{N}_1/\mathcal{N}$, and $N_0/N \approx \mathcal{N}_0/\mathcal{N}$, where $\mathcal{N}_0(\mathcal{N}_1)$ is the number of virtual agents choosing side 0 (1).

We now describe the *states* to be used in our ensemble average. It is convenient to begin by introducing *microstates* of the system. For each time step t, we define $a(k') \equiv n_1(k') - n_0(k')$, the accumulated difference between the number of times that after the appearance of string k', the resulting minority side was 1 ($n_1(k')$) or 0 ($n_0(k')$). A microstate μ is given by $(\overrightarrow{a}, k_0)$, where $\overrightarrow{a} \equiv (a(0), a(1), ..., a(\mathcal{H}-1))$ has the information of the net amount of virtual points assigned to the pool of strategies, and k_0 is the string containing the history that effectively showed up at time t.

The set of microstates is rather complex and very big. In order to get simpler expressions, we now turn back to the other approximation in which this work is based on, the period-two dynamics observed in [3]. Let us consider a game with $s = 2$, $m = 2$; in this case there are only four strings k : 00, 01, 10 and 11, representing all possible results for the last two minority groups. In their study of the time series of minority groups, Manuca et. al. kept a record of all the times when a given string k, for instance $k = 01$, appeared. This record shows a most remarkable behavior: odd occurrences of k are followed by a minority side χ

[1] Notice that this excludes the situation where two agents get the same pair. In any case, the probability that this happens in the game is vanishingly small even for a moderate value of m.

whose value is essentially random (e.g. $\chi_{odd} = 1$), while for the next (even) occurrence of k the result is deterministic, being the opposite of χ_{odd} ($\chi_{even} = 0$ in this example). This behavior was described as a period-two dynamics (PTD) by those authors, and can be seen to be essentially true for $m = 2$. In fact, we found that a plot of the probability that the data follows this rule, as a function of $z = 2^m/N$, displays scaling as can be seen in Fig.2, where one can also see that this probability is greater than 0.5 if $z \lesssim 0.7$.

The simultaneous use of the pool $\mathcal{L}$ *and* PTD allows a dramatic simplification of the ensemble of states. To understand this point, let us consider one step t of the game where a minority string k is followed by a minority side χ_t; as mentioned above, $\mathcal{L}/2$ strategies predict this output, and therefore get a point. Now, the next time that the string k appears, because of PTD, the *other* half of the strategies (predicting the opposite minority side) should get a point. Therefore, after an even appearance of k, *all* strategies will have one extra point. But remember that these points are assigned so that one can choose the more succesfull strategies, simply by picking up that with the greatest amount of points; in this sense, nothing should change if, rather than *adding* a point to the second half of strategies, we *remove* the point we already assigned to the first part. The implication of this procedure is great: we need to consider only those microstates where $|a(k)| \leq 1$. In the following, we will always refer to this subset of microstates, and the corresponding states.

A state is defined as the set of microstates that have the same value of the variables m, n, ϕ and p, where $n = \sum_{\{k'\}} \left|a(k^{'})\right|$, $\phi = |a(k_0)|$, and $p = a(k_0)$. As mentioned above, ϕ can take the values 0 or 1, while p can be equal to 0 or {1, -1}, respectively; $\{k^{'}\}$ is the set of all values of $k^{'}$. It is of some help to think of (ϕ, p) as an spin and its projection. In the future we will need to write down the number of microstates of the set, *i.e.* the degeneracy of the state, $g(n, m, \phi, p)$.

In Fig.3 we show a diagramatic representation of the microstates, that is also helpful to visualize the states. Each diagram has $\mathcal{H}$ rows, one for every possible history of m bits; the first column shows the value of $\overrightarrow{a}$ (the values 1, -1 and 0 are represented by an up or down arrow, and an empty site, respectively), and the second column simply displays the actual string k. There can only be one arrow for row, and up to $\mathcal{H}$ arrows in the diagram. Hence, the maximun number of points that the pool $\mathcal{L}$ can have is $2^{\mathcal{H}-1}\mathcal{H}$. [2]

[2] In the recent past there has been some discussion about the relevance of the memory [7] in the MG. By randomly generating strings of minority sides, Cavagna obtained results essentially equal to those of Fig.1; he concluded that memory is irrelevant in the MG, and should not be used to explain its behavior. On the contrary, Savit argued that "the dynamics and the information structure in the two versions are fundamentally the same". Our diagrams are particularly useful to shed some ligth on this point. To choose a string randomly, simply means to choose in this form the row in the *rigth* column of our diagrams; this eventually will change the 'microscopic' behavior of the model, but the register of what happens after the appearance of any history is not changed, because the rules that govern the *left* column of the diagram are not changed.

III. THE STANDARD DEVIATION

We now have completed the characterization of the states, and can come back to the calculation of $(\mathcal{N}_1 - \mathcal{N}_0)/\mathcal{N}$. As our approach is based on all the pairs of strategies (*i.e.* virtual agents) that can be formed out of the pool , there are three rather natural magnitudes to consider: $\mathcal{N}_{d1}$ ($\mathcal{N}_{d0}$) the number of agents whose response to a given microstate will be to choose $\chi = 1$ ($\chi = 0$) with certainty, hence the nickname 'decided' agents ; and $\mathcal{N}_u$, those agents that under the same circumstance can not make up their minds, thus the nickname 'undecided'. Clearly, $\mathcal{N} = \mathcal{N}_{d1} + \mathcal{N}_{d0} + \mathcal{N}_u$. In fact, after establishing a method to know how $\mathcal{N}_u$ will split into those choosing 0 ($\mathcal{N}_{u0}$) or 1 ($\mathcal{N}_{u1}$), we can write $\mathcal{N}_1 - \mathcal{N}_0 = \mathcal{N}_{d1} + \mathcal{N}_{u1} - \mathcal{N}_{d0} - \mathcal{N}_{u0}$.

Recalling that every possible pair of strategies corresponds to one virtual agent, to obtain the $\mathcal{N}'s$ is convenient to know $E_1^x(\overrightarrow{a}, k)$ and $E_0^x(\overrightarrow{a}, k)$, the number of strategies with x virtual points predicting $\chi = 1$ and $\chi = 0$, respectively, as the following minority side after the string k. Using E_1^x and E_0^x as a shortand notation for $E_1^x(\overrightarrow{a}, k)$ and $E_0^x(\overrightarrow{a}, k)$, it is

$$
\begin{aligned}
\mathcal{N}_u &= \sum_x E_1^x E_0^x \\
\mathcal{N}_{d1} &= \sum_x \sum_{j<x} E_1^x E_0^j + \frac{1}{2} \sum_x \sum_{j \neq x} E_1^x E_1^j + \sum_x (E_1^x)^2 \\
\mathcal{N}_{d0} &= \sum_x \sum_{j<x} E_1^j E_0^x + \frac{1}{2} \sum_x \sum_{j \neq x} E_0^x E_0^j + \sum_x (E_0^x)^2
\end{aligned} \tag{2}
$$

The last two terms of $\mathcal{N}_{d1}$ and $\mathcal{N}_{d0}$ are equal to $C_2^{\mathcal{L}/2} + \mathcal{L}/2$ (where $C_w^q \equiv \binom{w}{q}$), and are related with those agents for which both of their strategies predict the same side for the string k. As we are interested in the difference $(\mathcal{N}_1 - \mathcal{N}_0)$, these factors will cancel out, and don't need to be considered in the following.

Microstates $\mu_1 = (\overrightarrow{a}_1, k_1)$, $\mu_2 = (\overrightarrow{a}_2, k_2)$ obtained one from the other by interchanging one or more rows in the diagrams of Fig.(3) are symmetrical, in the sense that they have the same values E_i^x: $E_i^x(\mu_2) = E_i^x(\mu_1)$, ($i = 0$ or 1).This, in turn, implies $\mathcal{N}_u(\mu_2) = \mathcal{N}_u(\mu_1)$, and similarly for all terms of Eq.(2). In other words, the values of these magnitudes depend only on (m, n, ϕ, p), i.e are state dependent. Therefore, we will only need to consider the $E's$ for different *states,* together with the corresponding degeneracies $g(m, n, \phi, p)$. Thus, the expression for the standard deviation becomes

$$
\sigma^2 = \frac{N^2}{\mathcal{N}^2} \frac{1}{\Omega} \sum_{\{m,n,\phi,p\}} g(m, n, \phi, p) \left[\mathcal{N}_{d1} + \mathcal{N}_{u1} - \mathcal{N}_{d0} - \mathcal{N}_{u0}\right]^2 \tag{3}
$$

where $\Omega = \sum_{\{m,n,\phi,p\}} g(m, n, \phi, p)$.

For every state, we assume that each undecided agent chooses randomly between $\chi = 0$ and $\chi = 1$. Moreover, σ^2 will be averaged over a certain number of independent runs; hence, we only need the average values of $\mathcal{N}_{u1}$ and $\mathcal{N}_{u1}^2$, that are given by the standard expressions of the random walk, $\langle \mathcal{N}_{u1} \rangle = \frac{1}{2} \mathcal{N}_u$ and $\langle \mathcal{N}_{u1}^2 \rangle = \frac{1}{4} \mathcal{N}_u + \frac{1}{4} \mathcal{N}_u^2$.

Then

$$\sigma^2 = \frac{N^2}{\mathcal{N}^2}\frac{1}{\Omega}\sum_{\{m,n,\phi,p\}} g(m,n,\phi,p)\left[\mathcal{N}_u + (\mathcal{N}_{d1}-\mathcal{N}_{d0})^2\right] \tag{4}$$

The calculation of the different terms of Eq.(2), and the corresponding calculation of σ (Eq.(4)) is straigthforward, but rather lengthy and cumbersome. We have displaced to an Appendix some of the details; on the other hand it is useful to use a graphic representation to find E_0^x, E_1^x , and show how to obtain these terms. Notice that, regardless of which microstate we are looking at, the set of *all* possible strategies will always split into two groups of $\mathcal{L}/2$ strategies each, predicting $\chi = 0$ or 1 respectively; therefore, we just have to find how each one of these groups will split into smaller subgroups characterized by the number of points. Let us consider, to be specific, the case $m = 2$. In this case, we only need to know the E's for ($n = 0$ to 3, $\phi = 0$, $p = 0$), and for ($n = 1$ to 4, $\phi = 1$, $p = \pm 1$). In Fig.(4) we show these values for the case $\phi = 1$, $p = -1$, for strategies predicting $\chi = 0$ as the following minority side. Being $\phi = 1$ implies that we have to consider microstates having $a(k_0) \neq 0$. The simplest form to find the E's in this case, is by using the diagrams with one to four arrows. Thus, it is clear that there are $\mathcal{L}/2$ strategies with one point (*and* predicting $\chi = 0$). With $|\overrightarrow{a}| = 2$ (i.e. $n = 2$), there will be $\mathcal{L}/4$ strategies predicting only one of the minority sides (having one point), and $\mathcal{L}/4$ predicting both minorities correctly (thus having two points); analogously, for $n = 3$ there will be three groups of $\mathcal{L}/8$, $\mathcal{L}/4$, and $\mathcal{L}/8$ strategies, with 1, 2 and 3 points, respectively; finally, for $n = 4$ there are four groups with $\mathcal{L}/16$, $\mathcal{L}(1+2)/16$, $\mathcal{L}(1+2)/16$ and $\mathcal{L}/16$, with 1, 2, 3 and 4 points respectively. Graphically, this gives rise to a tree, as that shown in Fig.4, in which every row corresponds to a given value of n, and in every site of the tree one writes E_1^x. In fact, introducing e_i^x by writing $E_i^x(m,n,\phi,p) = 2^{\mathcal{H}-n+\phi-1}e_i^x(m,n,\phi,p)$, with $i = 1$ or 0, it is possible to see that the tree associated with e_i^x turns out to be Tartaglia's triangle, whose analytic properties are well known. The cases including states where ($n = \phi = 0$) can be handled in the same form. The difference between states with $\phi = 0$ and $\phi = 1$ it is apparent in the schemes (a) and (c) of Fig.3, but from the point of view of the trees, it simply implies that when $\phi = 0$ the values of n are 'shifted'; thus the 'upper' row has $n = 0$, and the 'lowest' one has $n = \mathcal{H} - 1$; moreover in row n, say, the different terms will have 0,1,...,n points. All terms needed to calculate σ are simply related with different sums and products of terms of these trees, without mixing factors from different rows.

To carry out the calculation of σ^2 we use $\mathcal{N}_u$, $|\mathcal{N}_{d1} - \mathcal{N}_{d0}|$ and g, as obtained in the Appendix. The actual expressions for $\mathcal{N}_u$ and $|\mathcal{N}_{d1} - \mathcal{N}_{d0}|$ are functions of the value of ϕ, namely

$$\begin{aligned} \mathcal{N}_u &= \left(2^{\mathcal{H}-n-1}\right)^2 \; C_{2n}^n \\ \mathcal{N}_{d1} - \mathcal{N}_{d0} &= 0 \end{aligned} \tag{5}$$

for $\phi = 0$, and

$$\begin{aligned} \mathcal{N}_u &= C_{2n-2}^n \qquad \text{if} \quad n > 1 \\ \mathcal{N}_u &= 0 \qquad\quad\;\; \text{if} \quad n = 1 \\ |\mathcal{N}_{d1} - \mathcal{N}_{d0}| &= C_{2n-2}^n + C_{2n-2}^n \end{aligned} \tag{6}$$

for $\phi = 1$.

The state degeneracy is [3]

$$g(m,n,\phi,p) = C_{\mathcal{H}}^{n-\phi}(\mathcal{H}-n+\phi)\ 2^{n-\phi} \tag{7}$$

where $\phi \leq n \leq \mathcal{H}+\phi-1$

Two points are worth mentioning, in relation with σ^2. First, using Eq.(A2) in Eq.(4), we verify that σ^2 can be written just in terms of $\mathcal{N}_u$'s factors. Furthermore, this expression can be factored out as follows: $\sigma^2 = N\ \mathcal{F}(m)$. Notice that our general assumptions require to consider N big enough, so that the replacement of the samples by the whole pool makes sense; hence, the factorization should also be valid in the same limit.

Our results are presented in Fig.5. We have done numerical simulations of the MG for N from 101 up to 1001, 50000 time steps, and $m = 2$ up to 13; the results where averaged over 32 independent runs. We show both the usual simulation data, as well as the results of the calculation with the equations derived in this work. As it can be seen, if $\frac{\sigma^2}{N} \geq 1$ (and $2^m/N \leq 0.1$) there is an excellent agreement between these two sets of data. For $2^m/N$ between 0.1 and 0.7 we only have a qualitative agreement, and for $2^m/N > 0.7$ our results (not shown) are clearly inadequate.This coincides with the range of validity of the results of Manuca et al. [3], i.e we only cover the informationally efficient phase. It is to be noticed that in the first region our results can be fitted to a high degree of accuracy by a straight line with slope -1.

IV. CONCLUSIONS

We have made a mean-field-like calculation of σ, the standard deviation of the MG. Our approach is based on two approximations: on the one hand, rather than using the properties of the actual samples used to calculate, we consider the whole pool of strategies, $\mathcal{L}$; in this form we benefit from the symmetry of the histories. It should be noticed that we actually quote results that are averages over several independent runs, so that it is reasonable to expect that they should be near to the symmetry of the pool. On the other hand, we considered that the period-two dynamics introduced by Manuca et.al [3] is correct within the range of validity of our work. We used these approximations to replace the time averages appearing in Eq.(1), by ensemble averages over the states.

Our results show an excellent agreement with data from the simulation in the region $\sigma^2/N \gtrsim 1$. The calculation embraces an important part of the informationally efficient region, but the agreement is lost when σ^2/N is near of its minimun value.

This method can also be useful to deal with cases where $s = 3$ (one needs to consider all the sets of three strategies) or greater.

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[3]in fact this expression is totally correct in the case where the microstates have a uniform probability distribution; in a game with memory the succession of strings is obviously concatenated and Eq.(7) is only approximated

APPENDIX:

We provide some extra details about the calculation of $\mathcal{N}_u$, $|\mathcal{N}_{d1} - \mathcal{N}_{d0}|$ and g.

The values of e_i^x, (for $i = 0, 1$), that we have already identified with the coeficients in Tartaglia's triangle, are state dependant (remember that in site (q, w), the coeficient of the tree is the combinatorial $C_w^q \equiv \binom{w}{q}$).

$$e_0^x(m,n,0,0) = e_1^x(m,n,0,0) = C_n^x \tag{A1a}$$

$$e_0^x(m,n,1,1) = e_1^x(m,n,1,-1) = C_{n-1}^x \tag{A1b}$$

$$e_0^x(m,n,1,-1) = e_1^x(m,n,1,1) = C_{n-1}^{x-1} \tag{A1c}$$

where $0 \leq x \leq n$, $0 \leq x \leq n-1$, and $1 \leq x \leq n$ in Eq.(A1a-c), respectively. In the first equation $\phi = 0$ and therefore $0 \leq n \leq \mathcal{H} - 1$, while the last two equations correspond to states with $\phi = 1$, then $1 \leq n \leq \mathcal{H}$.

We begin with $\phi = 0$ states, i.e. $(m, n, 0, 0)$ states, for which, following Eq.(2)

$$\mathcal{N}_u = 2^{2(\mathcal{H}-n-1)} \sum_{x=0}^{n} e_1^x e_0^x = \sum_{x=0}^{n} (C_n^x)^2 = C_{2n}^n$$

Moreover, for the same states

$$\mathcal{N}_{d1} = 2^{2(\mathcal{H}-n-1)} \sum_{x=0}^{n} \sum_{j=0}^{n} e_1^x e_0^j = 2^{2(\mathcal{H}-n-1)} \sum_{x=0}^{n} \sum_{j=0}^{n} C_n^x C_n^j = \mathcal{N}_{d0}$$

Therefore, $|\mathcal{N}_{d1} - \mathcal{N}_{d0}| = 0$.

Let us now consider $\phi = 1, p = 1$ states, i.e. $(m, n, 1, 1)$ states

$$\mathcal{N}_u = 2^{2(\mathcal{H}-n)} \sum_{x=1}^{n-1} e_1^x e_0^x = 2^{2(\mathcal{H}-n)} \sum_{x=1}^{n-1} C_{n-1}^x C_{n-1}^{x-1} = C_{2n-2}^n$$

for $n > 1$. If $n = 1$, it is $\mathcal{N}_u = 0$.

$\mathcal{N}_{d1}$ and $\mathcal{N}_{d0}$ are given by

$$\mathcal{N}_{d1} = 2^{2(\mathcal{H}-n)} \sum_{j<x} e_1^x e_0^j = 2^{2(\mathcal{H}-n)} \sum_{j=0}^{n-1} \sum_{x=j+1}^{n} C_{n-1}^j C_{n-1}^{x-1}$$

$$\mathcal{N}_{d0} = 2^{2(\mathcal{H}-n)} \sum_{x<j} e_1^x e_0^j = 2^{2(\mathcal{H}-n)} \sum_{x=1}^{n-2} \sum_{j=x+1}^{n-1} C_{n-1}^{x-1} C_{n-1}^j$$

Finally,

$$|\mathcal{N}_{d1} - \mathcal{N}_{d0}| = \sum_{j=0}^{n-1} C_{n-1}^{j} C_{n-1}^{j} + \sum_{j=0}^{n-1} C_{n-1}^{j} C_{n-1}^{j+1} = C_{2(n-1)}^{n-1} + C_{2n-2}^{n}$$

More explicitely,

$$|\mathcal{N}_{d1}(m,n,1,p) - \mathcal{N}_{d0}(m,n,1,p)| = \mathcal{N}_u(m,n-1,0,0) + \mathcal{N}_u(m,n,1,p) \tag{A2}$$

It is remarkable that the difference $|\mathcal{N}_{d1} - \mathcal{N}_{d0}|$ can be written just in terms of $\mathcal{N}_u$.

Notice that this 'nasty' collection of coefficients can be identified with combinations of elements of the trees. The simplest case is $\mathcal{N}_u$ in Eq.(2); in this case both trees (for $\chi = 0$ or 1) are equal, and $\mathcal{N}_u$ is given by the sum of the squares of the factors in row n of the tree.

To calculate the degeneracies $g(m,n,\phi,p)$, one has to count the number of microstates sharing the same values of (m,n,ϕ,p), using the diagramatic representation of the states. For $\phi = 0$,

$$g(m,n,0,0) = C_{\mathcal{H}}^{n}\ (\mathcal{H} - n)\ 2^n \tag{A3}$$

where $0 \leq n \leq \mathcal{H} - 1$. The first factor counts the number of ways that n arrows can be distributed into $\mathcal{H}$ places, the second is the number of strings available for being the actual history ($\times$ in the diagrams), and the third takes into account that each arrow can be up or down.

Analogously, we can write g for the case $\phi = 1$

$$g(m,n,1,\pm 1) = C_{\mathcal{H}}^{n}\ n\ 2^{n-1} \tag{A4}$$

with $1 \leq n \leq \mathcal{H}$.

It is possible to write these expressions in a form valid for both values of ϕ, as in Eq.(7).

Finally, the sum over all the degeneracies is the total amount of microstates, $\Omega(m) = \sum_{\{m,n,\phi,p\}} g(m,n,\phi,p) = 3^{2^m} 2^m$.

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FIGURES

FIG. 1. Scaling of σ^2/N vs $z = 2^m/N$, for different values of m and N: ■ $N = 101$; ○ $N = 201$; ▲ $N = 301$; •$N = 401$; ▽ $N = 501$; × $N = 1001$. For each value of N and m there are 32 runs, each one of $T = 10000$ time steps; for $N = 1001$, each run has 50000 time steps.

FIG. 2. Validity of the period-two dynamics. Number of times in which the dynamics comes true, over the total number of times in which the corresponding history occurs for an even time. Data are for different values of N: ■ $N = 101$; ○ $N = 201$; ▲ $N = 301$; •$N = 401$; ▽ $N = 501$; × $N = 1001$. For each value of N and m there are 32 runs, each one of $T = 10000$ time steps; for $N = 1001$, each run has 50000 time steps.

FIG. 3. Diagrams used to represent the ensamble of microstates for $m = 2$. ¿From top to bottom, the rows correspond to the strings 00, 01, 10 and 11. The symbol × is used to indicate the actual history; the arrows are used to represent the virtual points assigned to the strategies. For every history with an arrow, there are $\mathcal{L}/2$ points assigned. a: $(2, 2, 0, 0)$; b : $(2, 2, 1, 1)$; c: $(2, 2, 1, 1)$; d: $(2, 4, 1, -1)$. Notice that b and c are two microstates corresponding to the same state.

FIG. 4. Array of strategies corresponding to states with $m = 2$, $\phi = 1$, and $p = -1$. In this tree there are represented the $\mathcal{L}/2 = 8$ strategies predicting $\chi = 0$ for the actual string. Each row of the tree is characterized by a value of n, beginning with $n = 1$, up to $n = 4$. In each site of the tree the number between brackets gives the amount of virtual points ($x = 1, .., n$) for the corresponding strategies. There is an analogous tree for the strategies predicting $\chi = 1$ for the same string. The difference of both trees is given only for the virtual points assigned; in the tree for $\chi = 1$ these values are displaced, so that each row begins with $x = 0$, and ends up with $x = n - 1$.

FIG. 5. Data of σ^2/N vs $z = 2^m/N$ for $N = 1001$, both for the numerical simulations and from our calculations. As before, the results are averaged over 32 runs of 50000 time steps each.

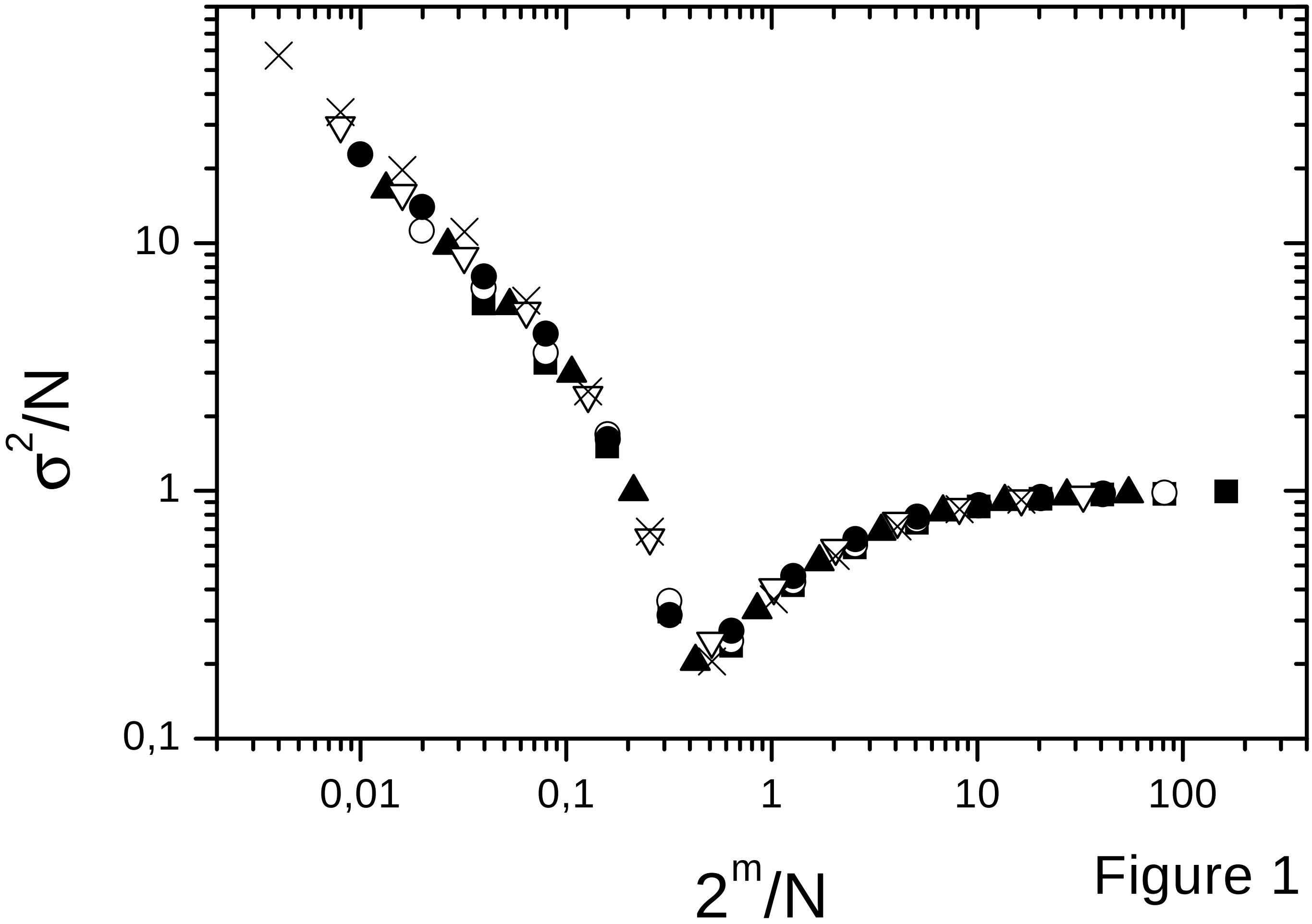

Figure 1

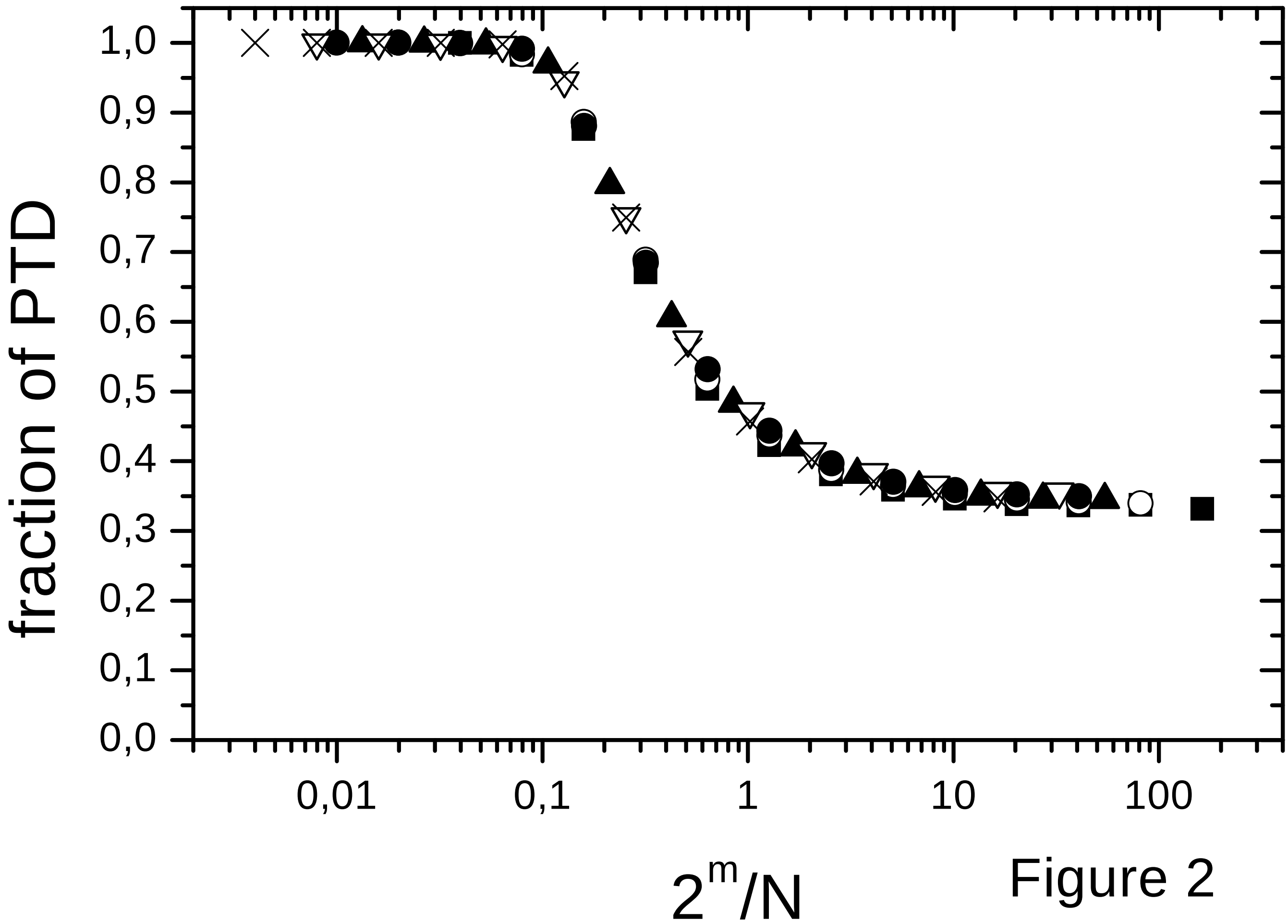

Figure 2

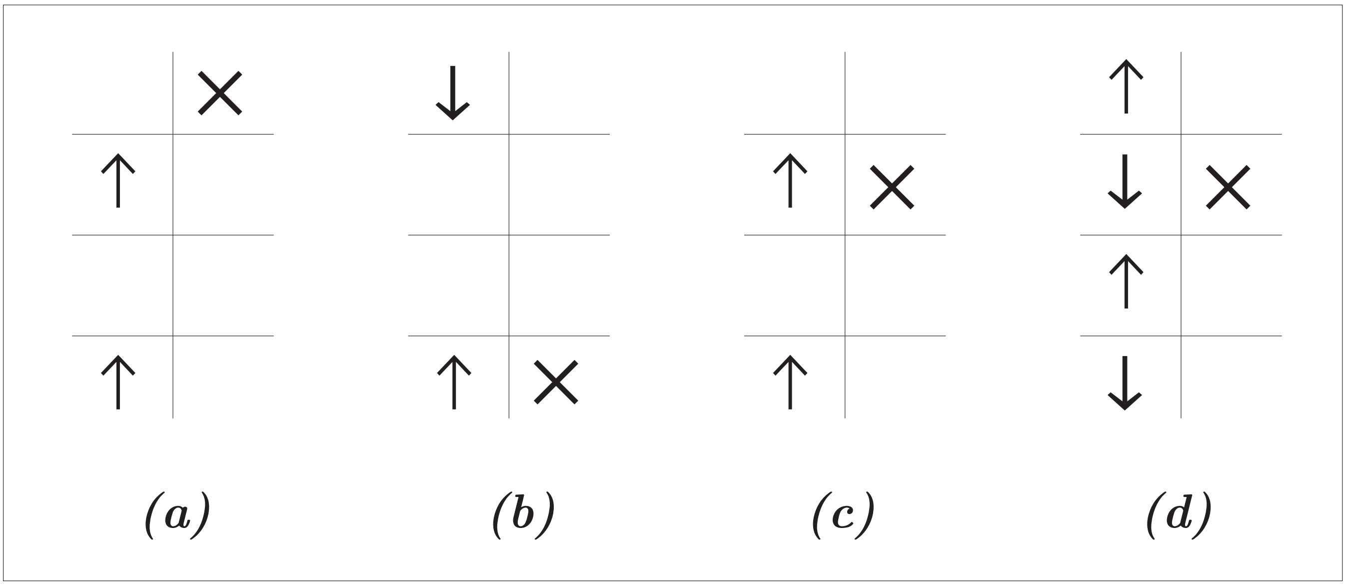

Figure 3

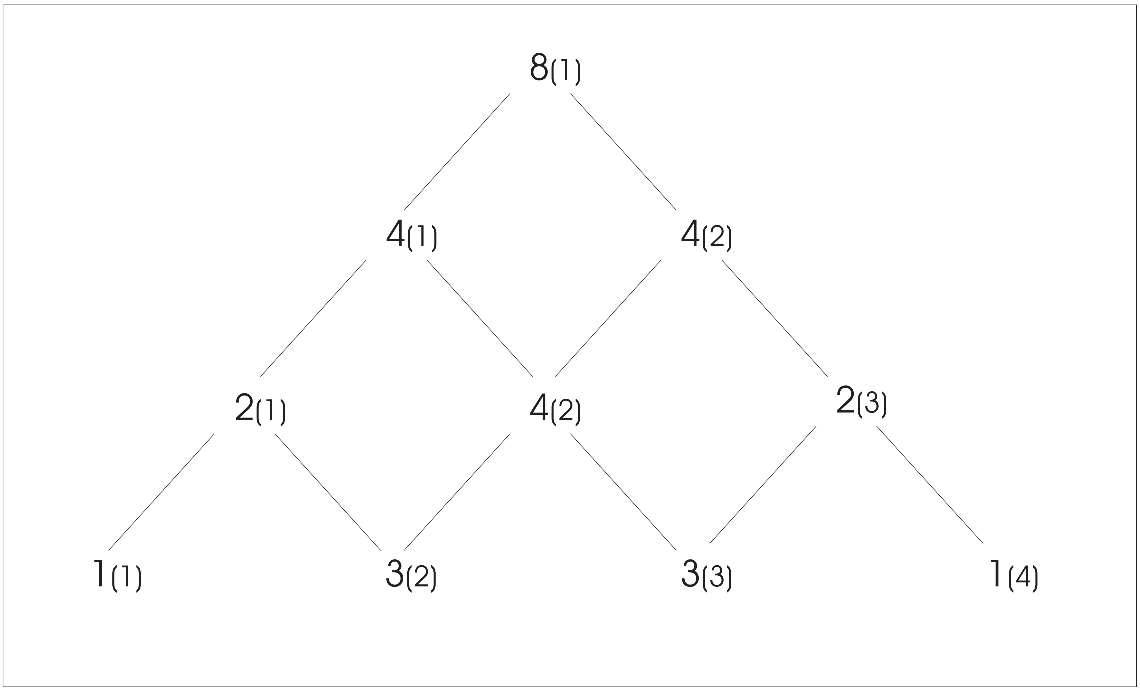

Figure 4

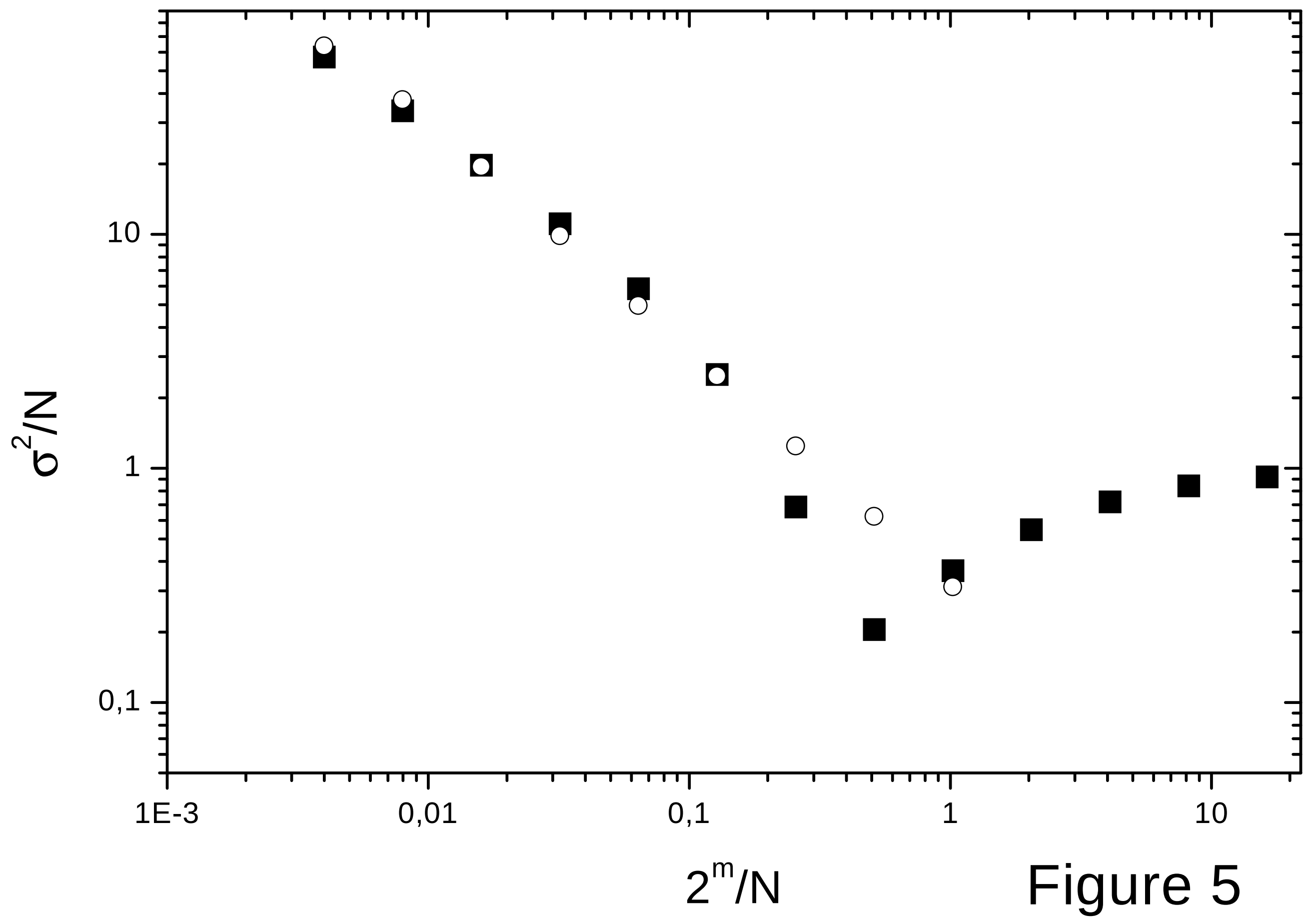

Figure 5